Set - Cover via Primal - Dual algorithm.

Primal:

$$\min c^T x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual

$$\max b^T y$$
$$A^T y \leq c$$
$$y \geq 0$$

$A \in \mathbb{R}^{m \times n}$

Note that: if x^* is optimal primal solution, $c^T x^* \leq OPT$. (integral optimal)

Relaxed CS conditions.

x, y feasible, $\exists\, \alpha, \beta \geq 1$ s.t.

$$y_i > 0 \Rightarrow \beta \cdot b_i \geq A_i^T x \geq b_i \qquad \text{— (I)}$$

$$x_j > 0 \Rightarrow c_j \geq \cancel{A_j^T}\, A^{j^T} y \geq \frac{c_j}{\alpha} \qquad \text{— (II)}$$

<u>Theorem</u>: If x, y are feasible and satisfy (I) & (II), then

$$c^T x \leq \alpha\beta\, b^T y \quad (\leq \alpha\beta \cdot c^T x^* = \alpha\beta \cdot OPT)$$

Proof skipped - use sequence of inequalities as for proving weak duality.

Approximation algorithm for set cover via Primal - Dual scheme.

Let f be max. frequency of any element: $\max_e |\{S : e \in S\}|$

Assume all costs are integers

$$\min \sum_S c_S x_S \qquad\qquad \max \sum_e y_e$$

$$\forall e, \ \sum_{S: e \in S} x_S \geq 1 \qquad\qquad \forall S, \ \sum_{e \in S} y_e \leq c_S$$

$$\forall S, \quad x_S \geq 0 \qquad\qquad \forall e \quad y_e \geq 0$$

Want an f-approx algo $\Rightarrow \alpha\beta = f$.

In fact, will obtain feasible x, y s.t.:

$$x_S > 0 \Rightarrow \sum_{e \in S} y_e = c_S \qquad (\alpha = 1)$$

$$y_e > 0 \Rightarrow f \geq \sum_{S: e \in S} x_S \geq 1$$

~~This the give~~ (and $x_S \in \{0,1\}$)

This will give us f-approx algo.

Algo:

dual vars

cover

Initially, all elements are uncovered, $y_e \leftarrow 0 \ \forall e$, $C \leftarrow \phi$

Until all elements are covered ($\cup\{S \in C\} \neq U$)

Pick an uncovered element $e \notin \cup\{S \in C\}$

Increase y_e until for some set S w/ $e \in S$, $\sum_{e' \in S} y_{e'} = c_S$

(set S becomes tight)

Add all tight sets to C, $x_S \leftarrow 1$

~~Claim: y is feasible throughout, x is feasible at end~~

Algo:

cover / uncovered elements

Initially, $x \leftarrow 0$, $y \leftarrow 0$, $C \leftarrow \phi$, $U' \leftarrow U$

while $U' \neq \phi$

Pick $e \in U'$

increase y_e until, for some S w/ $e \in S$, $\sum_{e' \in S} y_{e'} = c_S$

(set S becomes tight)

$C \leftarrow C \cup \{S : c_S = \sum_{e' \in S} y_{e'}\}$, $x_S = 1 \ \forall S \in C$

$U' \leftarrow U' \setminus \{e \in S : S \in C\}$

Note: (i) at beginning of loop, if $e \in U'$, then $y_{e'} = 0$

(ii) if $x_S = 1$ & $e \in S$, then y_e is ~~never~~ not increased

(iii) $x_S = 0$ or $x_S = 1$

Claim: x is feasible at end (easy)

y is feasible throughout (since for S, if at some point $\sum_{e \in S} y_e = c_S$, S is tight, all elements in S are covered, & y_e not raised for these elements afterwords).

Claim: x, y satisfy relaxed C-S.

Proof: If ~~a~~ set becomes tight, it is added to cover. Thus

$$x_S > 0 \Leftrightarrow \sum_{e \in S} y_e = c_S$$

Second condition is true by definition.

Set Cover via Randomized Rounding.

An f-approx algo: (deterministic rounding):

Solve LP:
$$\min \sum_S c_S x_S$$
$$\text{s.t. } \forall e, \sum_{S: e \in S} x_S \geq 1$$
$$x_S \geq 0$$

Let x^* be optimal solution.

$\forall S$; $x_S^* \geq \frac{1}{f}$, add S to cover. ($x_S = 1$)

This gives a set cover, since $\forall e, \sum_{S: e \in S} x_S^*$ has $\leq f$ terms, so at least some $x_S^* : e \in S$ must be $\geq \frac{1}{f}$.

Further $\sum_S c_S x_S \leq f \sum_S c_S x_S^* \leq f \cdot OPT$.

Randomized Rounding, $O(\log n)$-approx

Basic algo (will modify):

Let x^* be optimal (fractional) soln.

Add S to C (cover) w.p. x_S^* i.e., $x_S = 1$ w.p. x_S^*

Then $\mathbb{E}\left[\sum_{S \in C} c_S x_S\right] = OPT_f = \sum_{S \in C} c_S x_S^*$

Prob. this is a valid cover:

Fix e. The $Pr(e \text{ not covered}) = \prod_{S: e \in S} (1 - x_S^*)$, & $\sum_{S: e \in S} x_S^* \geq 1$.

$$\leq e^{-\sum_{S: e \in S} x_S^*} \leq \frac{1}{e}$$

Hen $Pr(e \text{ covered}) \geq 1 - \frac{1}{e}$

Now we pick ~~$2\ln n$~~ $2 \ln n$ such covers $C_1, C_2, \ldots, C_{2\ln n}$

expected cost $= 2 \ln n \cdot OPT_f$.

Pr for fixed element, $Pr(e \text{ not covered}) \leq \frac{1}{e^{2\ln n}} = \frac{1}{n^2}$

$\Rightarrow$ $Pr(C_1 \cup C_2 \cup \ldots C_{2\ln n}$ is a valid set cover$) \geq \frac{1}{n}$

Hence, $Pr(\text{failure}) = Pr(C_1 \cup \ldots \cup C_{2\ln n}$ is not a valid set cover OR cost $\geq 6 \ln n \cdot OPT_f) \leq \frac{1}{3} + \frac{1}{n} \leq \frac{1}{2}$

Just repeat a few times until we succeed.